JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 131, 392–403 (1988)

Directional Differentiability of Metric Projections onto Moving Sets at Boundary Points

ALEXANDER SHAPIRO

Department of Mathematics and Applied Mathematics, University of South Africa, P.O. Box 392, Pretoria, 0001 South Africa

Submitted by Frank H. Clarke

Received February 11, 1986

1. INTRODUCTION

Let X, Y be finite dimensional normed spaces and $\Omega: X \rightrightarrows Y$ be a multifunction assigning to each $x \in X$ a closed set $\Omega(x) \subset Y$. With $\Omega(x)$ are associated the distance function $d_\Omega: X \times Y \to \mathbb{R}$ and the set-valued metric projection $\Pi_\Omega: X \times Y \rightrightarrows Y$, which are defined as

$$d_\Omega(x, y) = \inf\{\|y - v\| : v \in \Omega(x)\}$$

and

$$\Pi_\Omega(x, y) = \{v \in \Omega(x): \|y - v\| = d_\Omega(x, y)\},$$

respectively. The aim of this paper is to investigate differential properties of Π_Ω in the vicinity of a point (x_0, y_0) with y_0 being a boundary point of the set $\Omega(x_0)$. We study Π_Ω in terms of a corresponding selection mapping $P_\Omega: X \times Y \to Y$, that is $P_\Omega(x, y) \in \Pi_\Omega(x, y)$ so that $v = P_\Omega(x, y)$ is a nearest to y point of $\Omega(x)$. It should be mentioned that since $y_0 \in \Omega(x_0)$, the set $\Pi_\Omega(x_0, y_0) = \{y_0\}$ is a singleton and hence P_Ω is uniquely defined at (x_0, y_0) with $P_\Omega(x_0, y_0) = y_0$.

In the case $\Omega(x) = \Omega_0$ is a constant (independent of x) nonempty convex set, differentiability properties of metric projections have been studied in a number of publications (see, e.g., [4–7, 17] and references therein). It was shown that if, in addition, the corresponding norm is strictly convex, then P_Ω is directionally differentiable at every point $y_0 \in \Omega_0$ [5, 11, 17]. The situation of moving set $\Omega(x)$ has been discussed in [9]. There differentiability of P_Ω has been investigated (by applying sensitivity analysis of nonlinear programs [3, 8]) essentially at a point y_0 outside the set $\Omega(x_0)$. In this paper we extend some results from [16], where necessary and in a sense sufficient conditions for directional differentiability of P_Ω at $y_0 \in \Omega_0$

0022-247X/88 $3.00

have been given for constant $\Omega(x)=\Omega_0$. Such an extension became possible owing to recent developments in the theory of point-to-set mappings. In particular, we make use of the concept of *pseudo-Lipschitzian* multifunctions [1, 15].

We suppose throughout that for a given point x_0 the set $\Omega(x_0)$ is nonempty and $y_0 \in \Omega(x_0)$. It will be shown that directional differentiability of P_Ω at (x_0, y_0) is determined by local behavior of the multifunction Ω and depends little on the chosen norm in Y. Assuming that Ω is pseudo-Lipschitzian we give in Section 2 necessary and sufficient conditions for the distance function $d_\Omega(x, y)$ to be directionally differentiable at (x_0, y_0). This will provide us with a basis for investigating differentiability properties of P_Ω in Section 3.

For a multifunction Ω we write gph Ω for its graph,

$$\operatorname{gph} \Omega = \{(x, y) : y \in \Omega(x)\}.$$

The multifunction Ω is called closed (convex) if gph Ω is a closed (convex) subset of $X \times Y$. (We use the norm $\|(x, y)\| = \|x\| + \|y\|$ on the product space $X \times Y$.) A multifunction $\Sigma : X \rightrightarrows Y$ is said to be positively homogeneous if $\Sigma(tx) = t\Sigma(x)$ for all $t > 0$ and $x \in X$. Clearly Σ is positively homogeneous if and only if its graph gph Σ is a cone. A (closed, convex) positively homogeneous multifunction is called a (closed, convex) *process* (cf. [14, Section 39]).

By $\operatorname{dist}(y, S)$ we denote the distance from a point y to set S,

$$\operatorname{dist}(y, S) = \inf\{\|y - v\| : v \in S\}.$$

Finally, a function $f(x)$ is said to be directionally differentiable at x_0 if the directional derivative

$$f'(x_0; y) = \lim_{t \to 0^+} \frac{f(x_0 + ty) - f(x_0)}{t}$$

exists for all y.

2. Directional Differentiability of the Distance Function

As a first step in our investigation we study differential properties of the distance function d_Ω. By convention $d_\Omega(x, y)$ is ∞ if the set $\Omega(x)$ is empty. Of course, for an arbitrary multifunction the associated distance function can behave quite irregularly. Therefore we restrict ourselves to multifunctions satisfying the pseudo-Lipschitzian condition which is defined as follows. Let us denote by $\delta(A; B)$ the diviation of a set A from set B,

$$\delta(A; B) = \sup\{\operatorname{dist}(y, B) : y \in A\}.$$

The multifunction $\Omega(x)$ is called *pseudo-Lipschitzian* at (x_0, y_0) if there exist neighbourhoods N_x of x_0, N_y of y_0 and a positive constant K such that

$$\delta(\Omega(x_1) \cap N_y;\, \Omega(x_2)) \leqslant K\, \|x_1 - x_2\|$$

for all $x_1, x_2 \in N_x$ (Aubin [1, p. 98]). The following result due to Rockafellar [15, Theorem 2.3] shows a natural interrelation between the above concept and continuity properties of d_Ω.

PROPOSITION 2.1. *The multifunction $\Omega(x)$ is pseudo-Lipschitzian at (x_0, y_0) if and only if the distance function d_Ω is Lipschitzian in a neighborhood of (x_0, y_0).*

In particular, the result above implies that if $\Omega(x)$ is pseudo-Lipschitzian at (x_0, y_0), then it is nonempty valued for all x in a neighborhood of x_0.

Now we introduce the following concept of cone approximation which will play an essential role in our considerations.

DEFINITION 1. We say that a subset S of a normed space is approximated at $z_0 \in S$ by a closed cone C, called an approximating cone, if

$$\lim_{\substack{z \to z_0 \\ z \in S}} \frac{\operatorname{dist}(z - z_0, C)}{\|z - z_0\|} = 0 \tag{2.1}$$

and

$$\lim_{\substack{z \to z_0 \\ z - z_0 \in C}} \frac{\operatorname{dist}(z, S)}{\|z - z_0\|} = 0. \tag{2.2}$$

The set $S - z_0$ and the approximating cone C are *tangent* (at zero) in the sense of Robinson [13, Definition 3]. A discussion of similar concepts and relevant references can be found in [12, Section 4].

It was shown in [16] that, in the finite-dimensional case, the approximating cone C exists if and only if the distance function $d_S(z) = \operatorname{dist}(z, S)$ is directionally differentiable at z_0, in which case $d'_S(z_0; v) = \operatorname{dist}(v, C)$. This implies that

$$C = \{v \colon d'_S(z_0; v) = 0\}$$

and hence once the approximating cone exists it is unique. These results settle the problem for constant multifunction Ω. In the following definition we extend the above concept of cone approximation by approximating Ω with a closed (not necessarily convex) process Σ.

DEFINITION 2. We say that Ω is tangentially differentiable at $(x_0, y_0) \in \operatorname{gph} \Omega$ if there exists a closed process $\Sigma: X \rightrightarrows Y$ such that

$$\lim_{\substack{(x, y) \to (x_0, y_0) \\ (x, y) \in \operatorname{gph} \Omega}} \frac{\operatorname{dist}(y - y_0, \Sigma(x - x_0))}{\|(x - x_0, y - y_0)\|} = 0 \tag{2.3}$$

and

$$\lim_{\substack{(x, y) \to (x_0, y_0) \\ (x - x_0, y - y_0) \in \operatorname{gph} \Sigma}} \frac{\operatorname{dist}(y, \Omega(x))}{\|(x - x_0, y - y_0)\|} = 0. \tag{2.4}$$

The process Σ is said to be a tangential approximation to Ω at (x_0, y_0) and is denoted $\Sigma = D\Omega(x_0, y_0)$.

Fixing $x = x_0$ in (2.3) and (2.4) one obtains that the set $\Omega(x_0)$ is approximated at y_0 by the cone $\Sigma(0)$. Furthermore, we have that

$$\operatorname{dist}(y, \Omega(x)) = \operatorname{dist}((x, y), \{x\} \times \Omega(x)),$$

and since $\{x\} \times \Omega(x)$ is a subset of $\operatorname{gph} \Omega$ it follows that

$$\operatorname{dist}(y, \Omega(x)) \geqslant \operatorname{dist}((x, y), \operatorname{gph} \Omega).$$

Therefore condition (2.4) implies (2.2) for $z_0 = (x_0, y_0)$, $S = \operatorname{gph} \Omega$ and $C = \operatorname{gph} \Sigma$. Similarly, (2.1) follows from (2.3). Consequently if the tangential approximation Σ exists, then the set $\operatorname{gph} \Omega$ is approximated at (x_0, y_0) by the cone $\operatorname{gph} \Sigma$. It may be noted that when it exists, the approximating cone coincides with the contingent (Bouligand) cone. Therefore for a *tangentially differentiable* multifunction Ω, $D\Omega(x_0, y_0)$ is equal to the "contingent derivative" introduced by Aubin [1, Definition 1].

The following example demonstrates that, in general, existence of cone approximation C to $\operatorname{gph} \Omega$ does not guarantee that the corresponding process Σ is the tangential approximation to Ω.

EXAMPLE. Let $f(x)$ be a piecewise linear function (of real variable) defined as follows. Consider numbers $a_k = k^{-1} - 2^{-k}$, $b_k = k^{-1} + 2^{-k}$ and let $f(x) = k^{-1} 2^k \min\{|x - a_k|, |x - b_k|\}$ for $x \in [a_k, b_k]$, $k = 6, 7, \ldots$, and $f(x)$ be zero otherwise. Consider the associated multifuncton $\Omega: \mathbb{R} \rightrightarrows \mathbb{R}$, $\Omega(x) = \{y: y \geqslant f(x)\}$, i.e., $\operatorname{gph} \Omega$ is the epigraph of f. It can be verified that the distance function corresponding to $\operatorname{gph} \Omega$ is directionally differentiable at $(0, 0)$ and hence the approximating cone exists and is given by $C = \{(x, y): y \geqslant 0\}$. However, the process Σ, with $\operatorname{gph} \Sigma = C$, is not tangent to Ω at $(0, 0)$.

A reason why the process Σ in the example above failed to provide the tangential approximation is that Ω was not pseudo-Lipschitzian. As we shall see later this does not happen for pseudo-Lipschitzian multifunctions.

Tangential differentiability of a multifunction is a natural generalization of the concept of directional differentiability. The following proposition makes this explicit.

PROPOSITION 2.2. *Let $\omega: X \to Y$ be a locally Lipschitz mapping. Then the corresponding (single-valued) multifunction $\Omega(x) = \{\omega(x)\}$ is tangentially differentiable at (x_0, y_0), $y_0 = \omega(x_0)$, if and only if ω is directionally differentiable at x_0.*

Proof. Suppose that ω is directionally differentiable at x_0. Then since ω is locally Lipschitz,

$$\omega(x_0 + v) - \omega(x_0) = \omega'(x_0; v) + o(\|v\|)$$

(e.g., [2, Lemma 3.2]). Now it can be easily seen that (2.3) and (2.4) hold for $\Sigma(v) = \{\omega'(x_0; v)\}$. Consequently, Ω is tangentially differentiable at (x_0, y_0).

On the other hand, suppose that Ω is tangentially differentiable at (x_0, y_0). Since $\Omega(x)$ is single-valued it follows from (2.4) that the tangential approximation $\Sigma(v)$ is also single-valued, say $\Sigma(v) = \{\sigma(v)\}$. From (2.3) we obtain

$$\omega(x_0 + v) - \omega(x_0) = \sigma(v) + o(\|(v, y - y_0)\|),$$

where $y = \omega(x)$. Since ω is locally Lipschitz this implies that

$$\omega(x_0 + v) - \omega(x_0) = \sigma(v) + o(\|v\|)$$

and the proof is complete. ∎

Now we formulate the main result of this section.

THEOREM 2.3. *Suppose that Ω is pseudo-Lipschitzian at (x_0, y_0). Then the following statements are equivalent:*

(i) *The set* gph Ω *is approximated at (x_0, y_0) by a closed cone C.*

(ii) *The multifunction Ω is tangentially differentiable at (x_0, y_0).*

(iii) *The distance function d_Ω is directionally differentiable at (x_0, y_0).*

Proof. Without loss of generality we can assume that $(x_0, y_0) = (0, 0)$. Suppose that condition (i) holds and consider the process Σ corresponding to the cone C, i.e., $\operatorname{gph} \Sigma = C$. We show that Σ gives the tangential approximation to Ω at (x_0, y_0).

First let us prove that the multifunction Σ is Lipschitzian. Let $x_1, x_2 \in X$, $x_2 \neq 0$, and $y_1 \in \Sigma(x_1)$. It follows from (2.2) that for all $t > 0$ there exists a vector $(\bar{x}_1(t), \bar{y}_1(t)) \in \operatorname{gph} \Omega$ such that

$$\|(tx_1, ty_1) - (\bar{x}_1(t), \bar{y}_1(t))\| = o(t). \tag{2.4}$$

Since Ω is pseudo-Lipschitzian we have that for sufficiently small t there is $y_2^*(t) \in \Omega(tx_2)$ such that

$$\|\bar{y}_1(t) - y_2^*(t)\| \leqslant K \|\bar{x}_1(t) - tx_2\|. \tag{2.5}$$

Furthermore, by (2.1) we can find $(\bar{x}_2(t), \bar{y}_2(t)) \in C$ such that

$$\|(tx_2, y_2^*(t)) - (\bar{x}_2(t), \bar{y}_2(t))\| = o(t). \tag{2.6}$$

Conditions (2.4)–(2.6) imply that

$$\|ty_1 - \bar{y}_2(t)\| \leqslant Kt \|x_1 - x_2\| + o(t) \tag{2.7}$$

and

$$\|tx_2 - \bar{x}_2(t)\| = o(t). \tag{2.8}$$

By the standard argument of compactness there exists a sequence $\{t_n\} \downarrow 0$ such that $t_n^{-1}\bar{y}_2(t_n)$ converges to a vector y_2. Also it follows from (2.8) that $t_n^{-1}\bar{x}_2(t_n) \to x_2$. Then since C is closed, $(x_2, y_2) \in C$. It only remains to note that because of (2.7)

$$\|y_1 - y_2\| \leqslant K \|x_1 - x_2\|.$$

Since x_1, x_2 and $y_1 \in \Sigma(x_1)$ are arbitrary this shows that the multifunction Σ is Lipschitzian. Of course, Lipschitz continuity of Σ implies that $\Sigma(x)$ is nonempty-valued for all x (cf. [1, Proposition 1]).

Now let us show that condition (2.3) holds. Consider a point $z = (x, y) \in \operatorname{gph} \Omega$. We have from (2.2) that there exists a point $\bar{z} = (\bar{x}, \bar{y}) \in \operatorname{gph} \Sigma$ such that $\|z - \bar{z}\| = o(\|z\|)$. Since Σ is Lipschitzian, there exists $y^* \in \Sigma(x)$ such that $\|y^* - \bar{y}\| \leqslant K \|x - \bar{x}\|$. This implies that $\|y - y^*\| = o(\|z\|)$ and hence (2.3) follows. Condition (2.4) can be proved in a similar way by using pseudo-Lipschitzian continuity of Ω. This completes the proof of the implication (i) $\Rightarrow$ (ii).

Now suppose that condition (ii) holds and let $\Sigma = D\Omega(x_0, y_0)$. It follows from the proof above that Σ is Lipschitzian. Since Σ is positively homogeneous the distance function

$$d_\Sigma(x, y) = \operatorname{dist}(y, \Sigma(x))$$

is also positively homogeneous. Therefore in order to prove that d_Ω is directionally differentiable it will be sufficient to show that

$$d_\Omega(x, y) = d_\Sigma(x, y) + o(\|(x, y)\|). \tag{2.9}$$

We have that $d_\Omega(x, y) = \|y - P_\Omega(x, y)\|$ and since d_Ω is Lipschitzian near $(0, 0)$, there exists a positive constant K such that

$$\|y - P_\Omega(x, y)\| \leqslant K\,\|(x, y)\|$$

for all (x, y) in a neighborhood of $(0, 0)$. Consequently,

$$\|P_\Omega(x, y)\| \leqslant K\,\|(x, y)\| + \|y\|$$

and hence

$$\|P_\Omega(x, y)\| \leqslant 2K\,\|(x, y)\|. \tag{2.10}$$

It follows from (2.3) that for every point (x, y) there exists a point $y^* = y^*(x, y) \in \Sigma(x)$ such that

$$\|P_\Omega(x, y) - y^*\| = o(\|(x, P_\Omega(x, y))\|).$$

By (2.10) we obtain that

$$\|P_\Omega(x, y) - y^*\| = o(\|(x, y)\|). \tag{2.11}$$

Now we have

$$d_\Sigma(x, y) \leqslant \|y - y^*\| \leqslant \|y - P_\Omega(x, y)\| + \|P_\Omega(x, y) - y^*\|.$$

Since $\|y - P_\Omega(x, y)\| = d_\Omega(x, y)$ and by (2.11) the last inequality implies that

$$d_\Sigma(x, y) \leqslant d_\Omega(x, y) + o(\|(x, y)\|).$$

The other inequality

$$d_\Omega(x, y) \leqslant d_\Sigma(x, y) + o(\|(x, y)\|)$$

can be proved in a similar way using Lipschitzian continuity of Σ. This proves the implication (ii) $\Rightarrow$ (iii).

Finally suppose that d_Ω is directionally differentiable at $(0, 0)$. Then since d_Ω is Lipschitzian near $(0, 0)$, the directional derivative $\alpha(x, y) = d'_\Omega((0, 0); (x, y))$ is Lipschitzian on $X \times Y$ and

$$d_\Omega(x, y) = \alpha(x, y) + o(\|(x, y)\|). \tag{2.12}$$

Let us define the cone C,

$$C = \{(x, y): \alpha(x, y) = 0\}.$$

Since α is Lipschitzian and hence continuous, the cone C is closed. We show that C is the approximating cone of the set $S = \operatorname{gph} \Omega$. Consider a point $(x, y) \in C$. By the definition of C and (2.12) we have $d_\Omega(x, y) = o(\|(x, y)\|)$. Since $d_\Omega(x, y) \geqslant \operatorname{dist}((x, y), S)$, condition (2.2) follows. Now consider the distance function

$$d_C(x, y) = \operatorname{dist}((x, y), C).$$

In order to prove (2.1) we have to show that for an arbitrary sequence $\{(x_n, y_n)\} \subset S$, $t_n^{-1} d_C(x_n, y_n)$ tends to zero as $t_n = \|(x_n, y_n)\| \to 0$. By the argument of compactness we can assume that $t_n^{-1}(x_n, y_n)$ converges to a vector $(\bar{x}, \bar{y})$. Since $(x_n, y_n) \in \operatorname{gph} \Omega$ we have that $d_\Omega(x_n, y_n) = 0$ and hence, by (2.12), $t_n^{-1} \alpha(x_n, y_n)$ tends to zero. Since α is positively homogeneous and continuous this implies that $\alpha(\bar{x}, \bar{y}) = 0$. Consequently $(\bar{x}, \bar{y}) \in C$ and thus $d_C(\bar{x}, \bar{y}) = 0$. Furthermore,

$$t_n^{-1} d_C(x_n, y_n) = d_C(t_n^{-1} x_n, t_n^{-1} y_n)$$

which by continuity of d_C implies that

$$t_n^{-1} d_C(x_n, y_n) \to d_C(\bar{x}, \bar{y}) = 0. \quad \blacksquare$$

It follows from the proof of Theorem 2.3 that if Ω is pseudo-Lipschitzian and tangentially differentiable at (x_0, y_0), then the process $\Sigma = D\Omega(x_0, y_0)$ is Lipschitzian and the directional derivatives of d_Ω are given by

$$d'_\Omega((x_0, y_0); (u, v)) = \operatorname{dist}(v, \Sigma(u)). \tag{2.13}$$

Now let $\Omega(x) = \Omega_0$ be constant. Then the distance function d_Ω is independent of x and is directionally differentiable at $y_0 \in \Omega_0$ if and only if there exists the approximating cone Σ_0 to Ω_0 at y_0 (see [16, Theorem 1]). Therefore in this case Ω is tangentially differentiable at (x_0, y_0), and $D\Omega(x_0, y_0) \equiv \Sigma_0$, if and only if the approximating cone Σ_0 exists.

One can find a thorough discussion of pseudo-Lipschitzian continuity in Rockafellar [15]. Some existence results for cone approximation are given in Robinson [13, Corollary 2] and Shapiro [16, Section 3]. Combined with Theorem 2.3 this provides sufficient conditions for directional differentiability of d_Ω for various constructions of Ω. The subject will be discussed further in the next section.

3. Directional Differentiability of Metric Projections

Consider a multifunction $\Omega(x)$ and let P_Ω be a corresponding selection metric projection. Suppose that Ω is tangentially differentiable at $(x_0, y_0) \in \operatorname{gph} \Omega$ and $\Sigma = D\Omega(x_0, y_0)$. Let us consider the associated point-to-set metric projection Π_Σ: $X \times Y \rightrightarrows Y$,

$$\Pi_\Sigma(x, y) = \{v \in \Sigma(x) \colon \|y - v\| = \operatorname{dist}(y, \Sigma(x))\}.$$

Since Σ is positively homogeneous, Π_Σ also is. In the following theorem we give, in a sense, sufficient and necessary conditions for directional differentiability of P_Ω.

Theorem 3.1. *Suppose that Ω is pseudo-Lipschitzian and tangentially differentiable at (x_0, y_0). Then*

$$\lim_{\substack{t \to 0^+ \\ (u', v') \to (u, v)}} \frac{\operatorname{dist}(P_\Omega(x_0 + tu', y_0 + tv') - y_0, t\Pi_\Sigma(u, v))}{t} = 0 \tag{3.1}$$

for all (u, v). Conversely, if P_Ω is Lipschitzian near (x_0, y_0) and is directionally differentiable, then Ω is pseudo-Lipschitzian and tangentially differentiable.

Proof. Without loss of generality we can assume that $(x_0, y_0) = (0, 0)$. Let Ω be pseudo-Lipschitzian and $\Sigma = D\Omega(x_0, y_0)$. Then as we have shown in the proof of Theorem 2.3 there exists a positive constant M such that

$$\|P_\Omega(x, y)\| \leqslant M \, \|(x, y)\| \tag{3.2}$$

for all (x, y) in a neighborhood of $(0, 0)$. We have that $\Pi_\Sigma(0, 0) = \{0\}$ and hence for $(u, v) = (0, 0)$ the identity (3.1) follows from (3.2). Therefore we suppose subsequently that $(u, v) \neq (0, 0)$. Consider a sequence $\{(u_n, v_n)\}$ converging to (u, v) and $t_n \to 0^+$. It follows from (3.2) that the sequence $\{t_n^{-1} P_n\}$, with $P_n = P_\Omega(t_n u_n, t_n v_n)$, is bounded and hence by the argument of compactness we can assume that this sequence converges to a vector $\bar{v}$. Since the distance function $\operatorname{dist}(\cdot, \Pi_\Sigma(u, v))$ is continuous it follows that

$$t_n^{-1} \operatorname{dist}(P_n, t_n \Pi_\Sigma(u, v)) \to \operatorname{dist}(\bar{v}, \Pi_\Sigma(u, v)). \tag{3.3}$$

Furthermore, by the definition of Σ, i.e., condition (2.3), there exists $v_n^* \in \Sigma(t_n u_n)$ such that $\|P_n - v_n^*\| = o(t_n)$. It follows that $t_n^{-1} v_n^* \to \bar{v}$ and since Σ is closed we obtain that $\bar{v} \in \Sigma(u)$. Now we have from Theorem 2.3 that

$$\operatorname{dist}(y, \Omega(x)) = \operatorname{dist}(y, \Sigma(x)) + o(\|(x, y)\|).$$

Consequently,

$$\|t_n v_n - P_n\| = \operatorname{dist}(t_n v_n, \Sigma(t_n u_n)) + o(t_n). \tag{3.4}$$

Since the function d_Σ is positively homogeneous and continuous we obtain from (3.4) that

$$\|v - \bar{v}\| = \operatorname{dist}(v, \Sigma(u)).$$

Together with $\bar{v} \in \Sigma(u)$ this implies that $\bar{v} \in \Pi_\Sigma(u, v)$. Therefore the left side expression in (3.3) tends to zero and thus (3.1) follows.

Conversely, if P_Ω is Lipschitzian near (x_0, y_0), then d_Ω is also Lipschitzian and hence Ω is pseudo-Lipschitzian by Proposition 2.1. Moreover, directional differentiability of P_Ω implies directional differentiability of d_Ω and consequently Ω is tangentially differentiable by Theorem 2.3. ∎

It follows from (3.1) that if conditions of the first part of Theorem 3.1 hold and the set $\Pi_\Sigma(u, v) = \{z\}$ is a singleton for some (u, v), then for $t > 0$,

$$P_\Omega(x_0 + tu, y_0 + tv) = y_0 + tz + o(t). \tag{3.5}$$

Thus z is the directional derivative of P_Ω in the direction (u, v). Moreover, if $\Pi_\Sigma(x, y) = \{P_\Sigma(x, y)\}$ is single-valued for all (x, y), then (3.1) gives the directional derivative uniformly in (u, v), i.e.,

$$P_\Omega(x_0 + u, y_0 + v) = y_0 + P_\Sigma(u, v) + o(\|(u, v)\|). \tag{3.6}$$

Suppose that the multifunction Ω is convex. Then the distance function d_Ω is convex and hence is locally Lipschitz (e.g., [14, p. 86]). Consequently, Ω is pseudo-Lipschitzian by Proposition 2.1. Moreover, since $S = \operatorname{gph} \Omega$ is convex it has the approximating cone which is given by the tangent cone

$$C = cl \cup \{\lambda(S - z_0) \colon \lambda > 0\},$$

where $z_0 = (x_0, y_0) \in S$ (cf. [16]). It follows then from Theorem 2.3 that Ω is tangentially differentiable and $\Sigma = D\Omega(x_0, y_0)$ is a convex process determined by $\operatorname{gph} \Sigma = C$. If, in addition, the norm is strictly convex, then the metric projections P_Ω and P_Σ are uniquely defined. Consequently, in this case Theorem 3.1 implies the following corollary which is an extension of the corresponding result for constant multifunction $\Omega(x) = \Omega_0$ and convex Ω_0 [5, 11, 17].

COROLLARY 3.2. *Suppose that the multifunction Ω is convex and the norm is strictly convex. Then P_Ω is directionally differentiable and equality* (3.6) *holds.*

Now let us consider an important case where Ω is defined by constraints

$$\Omega(x)=\{y: g_i(x, y)=0,\ i=1, ..., q;\ g_i(x, y)\leqslant 0,\ i=q+1, ..., p\}.$$

Suppose that the constraint functions g_i, $i=1, ..., p$, are C^1-smooth on $X\times Y$ and that the Mangasarian–Fromovitz condition [10] holds at (x_0, y_0):

(i) The gradient vectors $\nabla_y g_i(x_0, y_0)$, $i=1, ..., q$, are linearly independent.

(ii) There exists a vector b such that

$$\begin{aligned} b^{\mathrm{T}}\nabla_y g_i(x_0, y_0)&=0, \qquad i=1, ..., q,\\ b^{\mathrm{T}}\nabla_y g_i(x_0, y_0)&<0, \qquad i\in I, \end{aligned}$$

where $I=\{j: g_j(x_0, y_0)=0,\ q+1\leqslant j\leqslant p\}$.

Then it was shown in [15, p. 875] that Ω is pseudo-Lipschitzian at (x_0, y_0). Moreover, the approximating cone C to $\operatorname{gph}\Omega$ exists and is given by

$$C=\{(x, y): x^{\mathrm{T}}\nabla_x g_i+y^{\mathrm{T}}\nabla_y g_i=0,\ i=1, ..., q,\ x^{\mathrm{T}}\nabla_x g_i+y^{\mathrm{T}}\nabla_y g_i\leqslant 0,\ i\in I\}$$

with all gradients taken at the point (x_0, y_0) (Robinson [13, p. 504, Corollary 2]). It follows that Ω is tangentially differentiable and $\Sigma=D\Omega(x_0, y_0)$ is a convex process, $\operatorname{gph}\Sigma=C$. Consequently, if in addition the norm is strictly convex, then P_Ω is directionally differentiable and (3.6) follows.

References

1. J.-P. Aubin, Lipschitz behaviour of solutions to convex minimization problems, *Math. Oper. Res.* **9** (1984), 87–111.
2. V. F. Demyanov and A. M. Rubinov, On quasidifferentiable mappings, *Math. Operationsforsch. Statist. Ser. Optim.* **14** (1983), 3–21.
3. A. V. Fiacco, "Introduction to Sensitivity and Stability Analysis in Nonlinear Programming," Academic Press, New York, 1983.
4. S. Fitzpatrick and R. R. Phelps, Differentiability of the metric projection in Hilbert space, *Trans. Amer. Math. Soc.* **270** (1982), 483–501.
5. A. Haraux, How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities, *J. Math. Soc. Japan* **29** (1977), 615–631.
6. J. B. Hiriart-Urruty, At what points is the projection mapping differentiable? *Amer. Math. Monthly* **89** (1982), 456–460.
7. R. B. Holmes, Smoothness of certain metric projections on Hilbert space, *Trans. Amer. Math. Soc.* **184** (1973), 87–100.

8. K. JITTORNTRUM, Solution point differentiability without strict complementarity in nonlinear programming, *Math. Programming Stud.* **21** (1984), 127–138.
9. K. MALANOWSKI, Differentiability with respect to parameters of solutions to convex programming problems, *Math. Programming* **33** (1985), 352–361.
10. O. L. MANGASARIAN AND S. FROMOVITZ, The Fritz John necessary optimality conditions in the presence of equality and inequality constraints, *J. Math. Anal. Appl.* **17** (1967), 37–47.
11. G. P. MCCORMICK AND R. TAPIA, The gradient projection method under mild differentiability conditions, *SIAM J. Control* **10** (1972), 93–98.
12. J.-P. PENOT, On the existence of Lagrange multipliers in nonlinear programming in Banach spaces, *in* Lecture Notes in Control and Information Sciences Vol. 30, pp. 89–104, Springer-Verlag, Berlin, 1981.
13. S. M. ROBINSON, Stability theory for systems of inequalities, Part II: Differentiable nonlinear systems, *SIAM J. Numer. Anal.* **13** (1976), 497–513.
14. R. T. ROCKAFELLAR, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1970.
15. R. T. ROCKAFELLAR, Lipschitzian properties of multifunctions, *Nonlinear Anal. Theory Methods Appl.* **9** (1985), 867–885.
16. A. SHAPIRO, On diffferentiability of metric projections in $\mathbb{R}^n$, 1: Boundary case, *Proc. Amer. Math. Soc.* **99** (1987), 123–128.
17. E. H. ZARANTONELLO, Projections on convex sets in Hilbert space and spectral theory, *in* "Contributions to Nonlinear Functional Analysis," pp. 237–424, Academic Press, New York, 1971.